Roller Coaster Physics
Lesson 9
Clothoid Loops

1. Suppose we want a Force Factor of 4g's at the bottom of the loop (point 1). Let's investigate how we can design the loop so that we don't fall off the track at the top (point 2). Suppose we make the radius at the top of the loop (r_2) less than the radius at the bottom of the loop (r_1).

   a. First, what would the velocity at Point 1 (v_1) be if the *FF* at point 1 is 4 ($FF_1 = 4$)?

$$FF_1 = \frac{v_1^2}{r_1 g} + 1, \text{ and if } FF_1 = 4 \text{ then}$$

$$\frac{v_1^2}{r_1 g} + 1 = 4$$

$$\frac{v_1^2}{r_1 g} = 4$$

$$v_1^2 = 3r_1 g$$

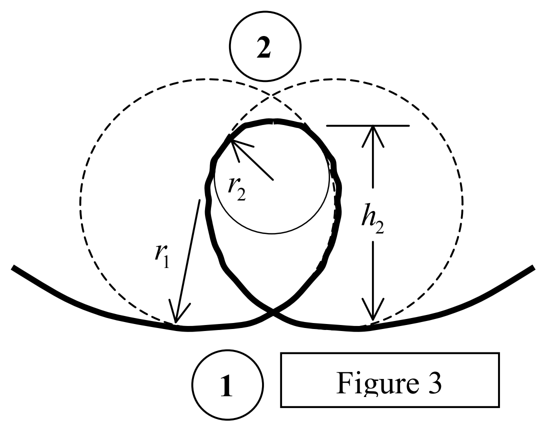


Figure 3

   b. Now, let's find what the velocity at the top of the loop (v_2) would be.

$$KE_1 + \cancel{PE_1}^{\,0} + \cancel{W_{1-2}}^{\,0} = KE_2 + PE_2$$

$$KE_1 = KE_2 + PE_2$$

$$\frac{1}{2}\cancel{m}v_1^2 = \frac{1}{2}\cancel{m}v_2^2 + \cancel{m}gh_2$$

$$v_1^2 = v_2^2 + 2gh_2, \qquad \text{but, from abve, } v_1^2 = 3r_1 g, \quad \text{so}$$

$$3r_1 g = v_2^2 + 2gh_2$$

$$v_2^2 = 3r_1 g - 2gh_2$$

   c. Again, let us assume a Force Factor of 0 at point 2. Let's find (v_2) using this fact.

$$FF_2 = \frac{v_2^2}{r_2 g} - 1, \text{ and if we let } FF_2 = 0, \text{ then}$$

$$\frac{v_2^2}{r_2 g} - 1 = 0$$

$$\frac{v_2^2}{r_2 g} = 1$$

$$v_2^2 = r_2 g$$

d. We have two expressions for $\left(v_2^{\,2}\right)$. Let's set them equal.

$$v_2^{\,2} = 3r_1 g - 2gh_2 \quad \text{and} \quad v_2^{\,2} = r_2 g, \text{ thus}$$
$$3r_1 \cancel{g} - 2\cancel{g}h_2 = r_2 \cancel{g}$$
$$3r_1 - 2h_2 = r_2$$

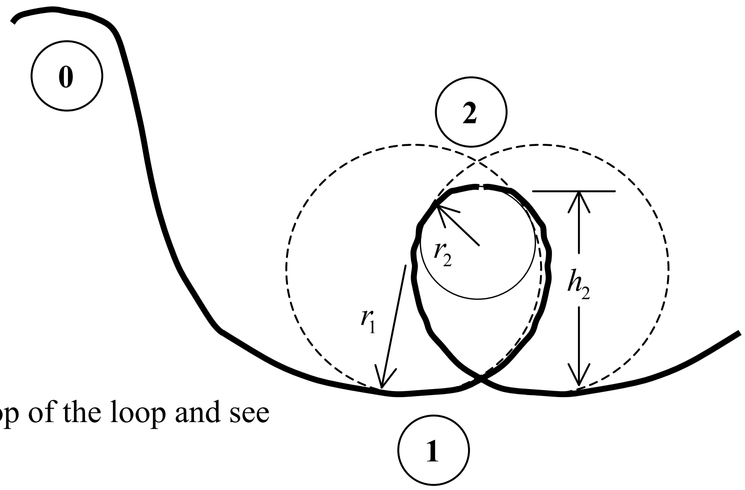
However, if you refer to Figure 3, you will notice that $h_2 = r_1 + r_2$. Therefore

$$3r_1 - 2h_2 = r_2$$
$$\text{and } h_2 = r_1 + r_2$$
$$\text{So, } \quad 3r_1 - 2(r_1 + r_2) = r_2$$
$$3r_1 - 2r_1 - 2r_2 = r_2$$
$$r_1 = 3r_2$$
$$r_2 = \frac{1}{3} r_1$$

Thus, if the radius of the top of the loop is approximately ⅓ the radius of the bottom of the loop, we can have a $FF_1 = 4$ with a $FF_2 = 0$.

e. Roller coasters today employ this type of loop. It is called a ***clothoid loop***. Circular loops cannot be used because they require greater entry speeds to complete the loop. The greater entry speeds subject passengers to greater centripetal acceleration through the lower half of the loop, therefore greater g's. If the radius is reduced at the top of the loop (approximately ⅓ the radius of the bottom of the loop), the centripetal acceleration is increased sufficiently to keep the passengers and the train from slowing too much as they move through the loop. A large radius is kept through the bottom half of the loop, thereby reducing the centripetal acceleration and the g's acting on the passengers.

2. **Example**. Let's design a coaster with a loop at the bottom of the first hill. The radius of track at the bottom is $r_1 = 45\,m$. The radius at the top is one third of this; $r_2 = 15\,m$. We want a Force Factor of 4 at point 1 and a Force Factor of 0 at point 2. What velocity at point 2 is required and how high must the first hill be to get this velocity?



$$FF_1 = 4 = \frac{v_1^2}{r_1 g} + 1$$

$$v_1^2 = 3 r_1 g = (3)(45m)\left(9.8\frac{m}{\sec^2}\right)$$

$$v_1^2 = 1323\frac{m^2}{\sec^2}$$

$$v_1 = 36.37\frac{m}{\sec}$$

Let's check the Force Factor at the top of the loop and see if it is 0 like we wanted.

$FF_2 = \dfrac{v_2^2}{r_2 g} - 1.$ But first we have to find v_2 :

$$KE_1 + \cancel{PE_1}^{0} + \cancel{W_{1-2}}^{0} = KE_2 + PE_2$$

$$\frac{1}{2}\cancel{m}v_1^2 = \frac{1}{2}\cancel{m}v_2^2 + \cancel{m}gh_2$$

$$v_1^2 = v_2^2 + 2gh_2$$

but, $h_2 \approx r_1 + r_2$

$$v_1^2 = v_2^2 + 2g(r_1 + r_2)$$

$$v_2^2 = v_1^2 - 2g(r_1 + r_2)$$

$$= 1323\frac{m^2}{\sec^2} - (2)\left(9.8\frac{m}{\sec^2}\right)(45m + 15m)$$

$$v_2^2 = 147\frac{m^2}{\sec^2}$$

$$v_2 = 12.12\frac{m}{\sec}$$

$$FF_2 = \frac{v_2^2}{r_2 g} - 1 = \frac{147\frac{m^2}{\sec^2}}{(15m)\left(9.8\frac{m}{\sec^2}\right)} - 1$$

$$= \frac{147\frac{m^2}{\sec^2}}{147\frac{m^2}{\sec^2}} - 1 = 1 - 1$$

$$FF_2 = 0$$

How high must the first hill be to give $v_1 = 36.37 \frac{m}{\sec}$?

Let's assume the top of the first hill is point 0 and that $v_0 = 2 \frac{m}{\sec}$.

$$KE_0 + PE_0 + \cancelto{0}{W_{0-1}} = KE_1 + \cancelto{0}{PE_1}$$

$$KE_0 + PE_0 = KE_1$$

$$\frac{1}{2} \cancel{m} v_0^2 + \cancel{m} g h_0 = \frac{1}{2} \cancel{m} v_1^2$$

$$v_0^2 + 2 g h_0 = v_1^2$$

$$h_0 = \frac{v_1^2 - v_0^2}{2g}$$

$$= \frac{1323 \frac{m^2}{\sec^2} - 4 \frac{m^2}{\sec^2}}{(2)\left(9.8 \frac{m}{\sec^2}\right)}$$

$$h_0 = 67.3\, m$$